Homework #8 of Topology II
Due Date: May 1, 2018

1. Let x, y be the standard coordinates and r, θ the polar coordinates on $\mathbb{R}^2 - 0$. (a) Show that the Poincare dual of the ray $\{(x, o)|x > 0\}$ in $\mathbb{R}^2 - 0$ is $d\theta/2\pi$ in $H^1(\mathbb{R}^2 - 0)$.

   (b) Show that the closed Poincare dual of the unit circle in $H^1(\mathbb{R}^2 - 0)$ is 0, but the compact Poincare dual is the nontrivial generator $\rho(r)dr$ in $H^1_c(\mathbb{R}^2 - 0)$ where $\rho(r)$ is a bump function with total integral 1.(A bump function is a smooth function whose support is contained in osme disc and whose graph looks like a "bump".)

2. (a) Show that there is a direct product decomposition

$$GL(n, \mathbb{R}) = O(n) \times \{\text{positive definite symmetric matrices}\}.$$

   (b) Use (a) to show that the structure group of any real vector bundle may be reduced to $O(n)$.

3. Compute $\mathrm{Vect}_k(S^1)$ the set of equivalence classes of isomorphic rank k vector bundles on S^1.

4. Use a Mayer-Vietoris argument as in proof of Thom isomorphism, show that if $\pi : E \to M$ is an orientable rank n bundle, then

$$H^*_c(E) \simeq H^{*-n}_c(M).$$

5. (Bott&Tu Page 70) (a) Show that if θ is the standard angle function on $\mathbb{R}^2$, measured in the counterclockwise direction, then $d\theta$ is positive on the circle S^1.

   (b) Show that if ϕ and θ are the spherical coordinate on S^2, the $d\theta \wedge d\phi$ is positive on sphere S^2.

6. (Bott&Tu Page 72) There exist 1-form ξ_α on U_α such that

$$\frac{1}{2\pi} d\phi_{\alpha\beta} = \xi_\beta - \xi_\alpha.$$

7. (Bott &Tu Page 77) Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and i the antipodal map on S^n:

$$i : (x_1, \cdots, x_{n+1}) \to (-x_1, \cdots, -x_{n+1}).$$

   The real projective space $\mathbb{R}P^n$ is the quotient of S^n by the equivalence relation

$$x \sim i(x), \text{ for } x \in \mathbb{R}^{n+1}.$$

(a) An invariant form on S^n is a form ω such that $i^*\omega = \omega$. The vector space of invariant forms on S^n, denoted by $\Omega^*(S^n)^I$, is a differential complex, and so the invariant cohomology is defined. Show that $H^*(\mathbb{R}P^n) = H^*(S^n)^I$.

(b) Show that the natural map $H^*(S^n)^I \to H^*(S^n)$ is injective.

(c) Give S^n the standard orientation (Page 70). Show that the antipodal map $i : S^n \to S^n$ is orientation-preserving for n odd and orientation-reversing for n even.

(d) Show that the de Rham cohomology of $\mathbb{R}P^n$ is

$$H^q(\mathbb{R}P^n) = \begin{cases} \mathbb{R} & \text{for } q = 0; \\ 0 & \text{for } 0 < q < n; \\ \mathbb{R} & \text{for } q = n \text{ odd}; \\ 0 & \text{for } q = n \text{ even}. \end{cases}$$